

# Limit theorems for decomposable branching processes in a random environment

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## Abstract

We study the asymptotics of the survival probability for the critical and decomposable branching processes in random environment and prove Yaglom type limit theorems for these processes. It is shown that such processes possess some properties having no analogues for the decomposable branching processes in constant environment

**Keywords** Decomposable branching processes; survival probability; random environment

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## 1 Introduction

The multitype branching processes in random environment we consider here can be viewed as a discrete-time stochastic model for the sizes of a geographically structured population occupying islands labelled  $0, 1, \dots, N$ . One unit of time represents a generation of particles (individuals). Particles located on island 0 give birth under influence of a randomly changing environment. They may migrate to one of the islands  $1, 2, \dots, N$  immediately after birth, with probabilities again depending upon the current environmental state. Particles of island  $i \in \{1, 2, \dots, N-1\}$  either stay at the same island or migrate to the islands  $i+1, 2, \dots, N$  and their reproduction laws are not influenced by any changing environment. Finally, particles of island  $N$  do not migrate and evolve in a constant environment.

The goal of this paper is to investigate the asymptotic behavior of the survival probability of the whole process and the distribution of the number

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of particles in the population given its survival or survival of particles of type 1.

Let  $m_{ij}$  be the mean number of type  $j$  particles produced by a type  $i$  particle at her death.

We formulate our main assumptions as

**Hypothesis A0 :**

- particles of type 0 form (on their own) a critical branching process in a random environment;
- particles of any type  $i \in \{1, 2, \dots, N\}$  form (on their own) a critical branching process in a constant environment, i.e.,  $m_{ii} = 1$ ;
- particles of any type  $i$  are able to produce descendants of all the next in order types (may be not as the direct descendants) but not any preceding ones. In particular,  $m_{ij} = 0$  for  $1 \leq j < i \leq N$  and  $m_{i,i+1} > 0$  for  $i = 1, \dots, N - 1$ .

Let  $X_n$  be the number of particles of type 0 and  $\mathbf{Z}_n = (Z_{n1}, \dots, Z_{nN})$  be the vector of the numbers of particles type 1, 2, ...,  $N$ , respectively, present at time  $n$ . Throughout of this paper considering the  $(N + 1)$ -type branching process it is assumed (unless otherwise specified) that  $X_0 = 1$  and  $\mathbf{Z}_0 = (0, \dots, 0) = \mathbf{0}$ .

We investigate asymptotics of the survival probability of this process as  $n \rightarrow \infty$  and the distribution of the number of particles in the process at moment  $n$  given  $Z_{n1} > 0$  or  $\mathbf{Z}_n \neq \mathbf{0}$ . Note that the asymptotic behavior of the survival probability for the case  $N = 1$  has been investigated in [9] under stronger assumptions than those imposed in the present paper. The essential novelty of this paper are Yaglom-type limit theorems for the population vector  $\mathbf{Z}_n$  (see Theorem 6 below).

The structure of the remaining part of this paper is as follows. In Section 2 we recall known facts for decomposable branching processes in constant environments and show some preliminary results. Section 3 deals with the  $(N + 1)$ -type decomposable branching processes in random environment. Here we study the asymptotic behavior of the survival probability and prove a Yaglom-type conditional limit theorem for the number of particles in the process given  $Z_{n1} > 0$ . In Section 4 we consider a 3-type decomposable branching process in random environment and, proving a Yaglom-type conditional limit theorem under the condition  $Z_{n1} + Z_{n2} > 0$ , show the essential difference of such processes with the decomposable processes evolving in constant environment.

## 2 Multitype decomposable branching processes in a constant environment

The aim of this section is to present a number of known results about the decomposable branching processes we are interesting in the case of a constant environment and, therefore, we do not deal with particles of type 0.

If Hypothesis A0 is valid then the mean matrix of our process has the form

$$\mathbf{M} = (m_{ij}) = \begin{pmatrix} 1 & m_{12} & \dots & \dots & m_{1N} \\ 0 & 1 & m_{23} & \dots & m_{2N} \\ 0 & 0 & 1 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & m_{N-1,N} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \quad (1)$$

where

$$m_{i,i+1} > 0, \quad i = 1, 2, \dots, N-1. \quad (2)$$

Under conditions (1) and (2) one obtains a complete ordering  $1 \longrightarrow 2 \longrightarrow \dots \longrightarrow N$  of types.

Observe that according to the classification given in [6] the process we consider is *strongly critical*.

In the sequel we need some results from [5] and [6]. To this aim we introduce additional notation.

1) For any vector  $\mathbf{s} = (s_1, \dots, s_p)$  (the dimension will usually be clear from the context), and integer valued vector  $\mathbf{k} = (k_1, \dots, k_p)$  define

$$\mathbf{s}^{\mathbf{k}} = s_1^{k_1} \dots s_p^{k_p}.$$

Further, let  $\mathbf{1} = (1, \dots, 1)$  be a vector of units and let  $\mathbf{e}_i$  be a vector whose  $i$ -th component is equal to one while the remaining are zeros.

2) The first and second moments of the components of the population vector  $\mathbf{Z}_n = (Z_{n1}, \dots, Z_{nN})$  will be denoted as

$$\begin{aligned} m_{il}(n) &:= \mathbf{E}[Z_{nl} | \mathbf{Z}_0 = \mathbf{e}_i], \quad m_{il} := m_{il}(1), \\ b_{ikl}(n) &:= \mathbf{E}[Z_{nk}Z_{nl} - \delta_{kl}Z_{nl} | \mathbf{Z}_0 = \mathbf{e}_i], \quad b_{ikl} := b_{ikl}(1). \end{aligned} \quad (3)$$

To go further we introduce probability generating functions

$$h^{(i,N)}(\mathbf{s}) := \mathbb{E} \left[ \prod_{k=i}^N s_k^{\eta_{ik}} \right], \quad 1 \leq i \leq N, \quad (4)$$

where  $\eta_{ij}$  represents the number of daughters of type  $j$  of a mother of type  $i \in \{1, 2, \dots, N\}$ . Let

$$H_n^{(i,N)}(\mathbf{s}) := \mathbb{E} \left[ \prod_{k=i}^N s_k^{Z_{nk}} | \mathbf{Z}_0 = \mathbf{e}_i \right], \quad 1 \leq i \leq N, \quad (5)$$

be the probability generating functions for the vector of the number of particles at moment  $n$  given the process is initiated at time 0 by a singly particle of type  $i \in \{1, 2, \dots, N\}$ . Clearly,  $H_1^{(i,N)}(\mathbf{s}) = h^{(i,N)}(\mathbf{s})$ . Denote

$$\begin{aligned} \mathbf{H}_n(\mathbf{s}) &: = \left( H_n^{(1,N)}(\mathbf{s}), \dots, H_n^{(N,N)}(\mathbf{s}) \right), \\ \mathbf{Q}_n(\mathbf{s}) &: = \left( Q_n^{(1,N)}(\mathbf{s}), \dots, Q_n^{(N,N)}(\mathbf{s}) \right) = \left( 1 - H_n^{(1,N)}(\mathbf{s}), \dots, 1 - H_n^{(N,N)}(\mathbf{s}) \right). \end{aligned}$$

As usually, for two sequences  $a_n, b_n$  we write  $a_n \sim b_n$ ,  $a_n = O(b_n)$ ,  $a_n = o(b_n)$  and  $a_n \asymp b_n$  meaning that these relationships are valid as  $n \rightarrow \infty$ . In particular,  $a_n \asymp b_n$  if and only if

$$0 < \liminf_{n \rightarrow \infty} a_n/b_n \leq \limsup_{n \rightarrow \infty} a_n/b_n < \infty.$$

The following theorem is a simplified combination of the respective results from [5] and [6]:

**Theorem 1** *Let  $\{\mathbf{Z}_n, n = 0, 1, \dots\}$  be a strongly critical multitype branching process satisfying (1) and (2). Then, as  $n \rightarrow \infty$*

$$m_{il}(n) \sim c_{il} n^{l-i}, \quad i \leq l, \quad (6)$$

where  $c_{il}$  are positive constants known explicitly (see [6], Theorem 1);

2) if  $b_{ikl} < \infty$ ,  $i, k, l = 1, \dots, N$  then

$$b_{ikl}(n) \sim c_{ikl} n^{k+l-2i+1}, \quad (7)$$

where  $c_{ikl}$  are constants known explicitly (see [6], Theorem 1) and

$$Q_n^{(i,N)}(\mathbf{0}) = 1 - H_n^{(i,N)}(\mathbf{0}) = \mathbf{P}(\mathbf{Z}_n \neq \mathbf{0} | \mathbf{Z}_0 = \mathbf{e}_i) \sim c_i n^{-2^{-(N-i)}}, \quad c_i > 0. \quad (8)$$

Let  $H(s_1, \dots, s_p) = H(\mathbf{s})$  be a multivariate probability generating function with

$$m_l := \frac{\partial H(\mathbf{s})}{\partial s_l} \Big|_{\mathbf{s}=\mathbf{1}}, \quad b_{kl} := \frac{\partial^2 H(\mathbf{s})}{\partial s_k \partial s_l} \Big|_{\mathbf{s}=\mathbf{1}} < \infty.$$

**Lemma 2** (see formula (1), page 189, in [3]) For any  $\mathbf{s} = (s_1, \dots, s_p) \in [0, 1]^p$  we have

$$\sum_{l=1}^p m_l (1 - s_l) - \frac{1}{2} \sum_{k,l=1}^p b_{kl} (1 - s_k) (1 - s_l) \leq 1 - H(\mathbf{s}) \leq \sum_{l=1}^p m_l (1 - s_l).$$

From now on we agree to denote by  $C, C_0, C_1, \dots$  positive constants which may be different in different formulas.

For  $s = (s_1, \dots, s_N)$  put

$$M_i(n; \mathbf{s}) := \sum_{l=i}^N m_{il}(n) (1 - s_l), \quad B_i(n; \mathbf{s}) := \frac{1}{2} \sum_{k,l=i}^N b_{ikl}(n) (1 - s_k) (1 - s_l). \quad (9)$$

**Lemma 3** Let the conditions of Theorem 1 be valid. Then for any tuple  $t_1, \dots, t_N$  of positive numbers and

$$1 - s_l = n^{-t_l}, \quad l = 1, 2, \dots, N$$

there exists  $C_+ < \infty$  such that, for all  $n = 1, 2, \dots$

$$Q_n^{(i,N)}(\mathbf{s}) \leq C_+ \min \left\{ n^{-2^{-(N-i)}}, n^{-\min_{i \leq l \leq N} (t_l - l + i)} \right\}.$$

If, in addition,

$$\min_{i \leq l \leq N} (t_l - l + i) \geq 1 \quad (10)$$

then there exists a positive constant  $C_-$  such that, for all  $n = 1, 2, \dots$

$$C_- n^{-\min_{i \leq l \leq N} (t_l - l + i)} \leq Q_n^{(i,N)}(\mathbf{s}) \leq C_+ n^{-\min_{i \leq l \leq N} (t_l - l + i)}. \quad (11)$$

PROOF Take  $\varepsilon \in (0, 1]$  and denote  $\mathbf{s}(\varepsilon) = (1 - \varepsilon n^{-t_1}, \dots, 1 - \varepsilon n^{-t_N})$ . By Lemma 2 and monotonicity of  $Q_n^{(i,N)}(\mathbf{s}(\varepsilon))$  in  $\varepsilon$ , we have

$$M_i(n; \mathbf{s}(\varepsilon)) - B_i(n; \mathbf{s}(\varepsilon)) \leq Q_n^{(i,N)}(\mathbf{s}(\varepsilon)) \leq Q_n^{(i,N)}(\mathbf{s}) \leq M_i(n; \mathbf{s}). \quad (12)$$

In view of (6) - (7) there exist positive constants  $C_j, j = 1, 2, 3, 4$  such that

$$\begin{aligned} \varepsilon C_1 n^{-\min_{i \leq l \leq N} (t_l - l + i)} &\leq \varepsilon C_1 \sum_{l=i}^N \frac{n^{l-i}}{n^{t_l}} \leq M_i(n; \mathbf{s}(\varepsilon)) = \varepsilon \sum_{l=i}^N m_{il}(n) n^{-t_l} \\ &\leq M_i(n; \mathbf{s}) \leq C_2 \sum_{l=i}^N \frac{n^{l-i}}{n^{t_l}} \leq C_3 n^{-\min_{i \leq l \leq N} (t_l - l + i)} \end{aligned} \quad (13)$$

and

$$0 \leq B_i(n; \mathbf{s}(\varepsilon)) \leq \varepsilon^2 C_4 \sum_{k,l=i}^N \frac{n^{k-i+1+l-i}}{n^{t_k} n^{t_l}}.$$

If now  $\min_{i \leq k \leq N} (t_k - k + i - 1) \geq 0$ , then for a fixed  $\varepsilon > 0$

$$0 \leq B_i(n; \mathbf{s}(\varepsilon)) \leq \varepsilon^2 C_4 \sum_{k,l=i}^N \frac{1}{n^{t_l - (l-i)} n^{t_k - (k-i+1)}} \leq \varepsilon^2 N^2 C_4 n^{-\min_{i \leq l \leq N} (t_l - l + i)}. \quad (14)$$

Take  $0 < \varepsilon < \min\{1, C_1/N^2 C_4\}$ . Then the estimates (12)–(14) give (11) with  $C_- = \varepsilon C_1 - \varepsilon^2 N^2 C_4$  and  $C_+ = C_3$ .  $\square$

Write  $\mathbf{0}^{(r)} = (0, 0, \dots, 0)$  and  $\mathbf{1}^{(r)} = (1, 1, \dots, 1)$  for the  $r$ -dimensional vectors all whose components are zeros and ones, respectively; set  $\mathbf{s}_r = (s_r, s_{r+1}, \dots, s_N)$  and denote by  $I\{\mathcal{A}\}$  the indicator of the event  $\mathcal{A}$ .

The next lemma, in which we assume that  $\mathbf{Z}_0 = \mathbf{e}_1$  gives an approximation for the function  $Q_n^{(1,N)}(\mathbf{0}^{(r)}, \mathbf{s}_{r+1})$ .

**Lemma 4** *If  $\min_{r+1 \leq l \leq N} (t_l - l + 1) > 2^{-(r-1)}$  and*

$$1 - s_l = n^{-t_l}, \quad l = r+1, r+2, \dots, N,$$

*then, as  $n \rightarrow \infty$*

$$Q_n^{(1,N)}(\mathbf{0}^{(r)}, \mathbf{s}_{r+1}) \sim \mathbf{P}(Z_{nr} > 0) \sim c_r n^{-2^{-(r-1)}}.$$

PROOF In view of (8) we have for  $\mathbf{s}_{r+1} \in [0, 1]^{N-r}$ :

$$\begin{aligned} \mathbf{P}(Z_{nr} > 0) &\leq \mathbf{P}(\cup_{j=1}^r \{Z_{nj} > 0\}) = Q_n^{(1,N)}(\mathbf{0}^{(r)}, \mathbf{1}^{(N-r)}) \\ &\leq Q_n^{(1,N)}(\mathbf{0}^{(r)}, \mathbf{s}_{r+1}) = \mathbf{E} \left[ 1 - s_{r+1}^{Z_{n,r+1}} \dots s_N^{Z_{nN}} I\{\cap_{j=1}^r \{Z_{nj} = 0\}\} \right] \\ &\leq \mathbf{P}(\cup_{j=1}^r \{Z_{nj} > 0\}) + \mathbf{E} \left[ 1 - s_{r+1}^{Z_{n,r+1}} \dots s_N^{Z_{nN}} \right] \\ &\leq \sum_{j=1}^r \mathbf{P}(Z_{nj} > 0) + \mathbf{E} \left[ 1 - s_{r+1}^{Z_{n,r+1}} \dots s_N^{Z_{nN}} \right] \\ &= (1 + o(1)) \mathbf{P}(Z_{nr} > 0) + Q_n^{(1,N)}(\mathbf{1}^{(r)}, \mathbf{s}_{r+1}). \end{aligned}$$

Further, by the conditions of the lemma we deduce

$$\begin{aligned} Q_n^{(1,N)}(\mathbf{1}^{(r)}, \mathbf{s}_{r+1}) &\leq \sum_{l=r+1}^N m_{1l}(n) n^{-t_l} \\ &\leq C n^{-\min_{r+1 \leq l \leq N} (t_l - l + 1)} = o(n^{-2^{-(r-1)}}). \end{aligned}$$

Hence the statement of the lemma follows.  $\square$

## 2.1 The case of two types

Here we consider the situation of two types and investigate the behavior of the function  $1 - H_n^{(1,2)}(s_1, s_2)$  as  $n \rightarrow \infty$  assuming that  $1 - s_i = n^{-t_i}$ ,  $i = 1, 2$ .

**Lemma 5** *If the conditions of Theorem 1 are valid for  $N = 2$ , then*

$$1 - H_n^{(1,2)}(s_1, s_2) \asymp \begin{cases} n^{-1/2} & \text{if } t_1 \in (0, \infty), 0 < t_2 \leq 1; \\ n^{-t_2/2} & \text{if } t_1 \in (0, \infty), 1 < t_2 < 2; \\ n^{-1} & \text{if } 0 < t_1 < 1, t_2 \geq 2; \\ n^{-1-\min(t_1-1, t_2-2)} & \text{if } t_1 \geq 1, t_2 \geq 2. \end{cases}$$

PROOF Observe that for any  $0 \leq s_1 \leq s'_1 \leq 1$

$$\begin{aligned} H_n^{(1,2)}(s'_1, s_2) - H_n^{(1,2)}(s_1, s_2) &= \mathbf{E} \left[ \left( (s'_1)^{Z_{n1}} - s_1^{Z_{n1}} \right) s_2^{Z_{n2}} \right] \\ &\leq \mathbf{E} \left[ 1 - s_1^{Z_{n1}} \right] = 1 - H_n^{(1,1)}(s_1) \\ &\leq \mathbf{P}(Z_{n1} > 0 | \mathbf{Z}_0 = \mathbf{e}_1) \leq Cn^{-1}. \end{aligned} \quad (15)$$

Let now  $m = m(s_2)$  be specified by the inequalities

$$Q_m^{(2,2)}(0) \leq 1 - s_2 = n^{-t_2} \leq Q_{m-1}^{(2,2)}(0). \quad (16)$$

In view of

$$Q_m^{(2,2)}(0) = 1 - H_m^{(2,2)}(0) = \mathbf{P}(Z_{m2} > 0 | \mathbf{Z}_0 = \mathbf{e}_2) \sim \frac{2}{m \text{Var} \eta_{22}},$$

it follows that  $m \sim 2n^{t_2}/\text{Var} \eta_{22}$ . Using this fact, estimate (15) and the branching property

$$H_n^{(1,2)}(H_m^{(1,2)}(\mathbf{s}), H_m^{(2,2)}(s_2)) = H_{n+m}^{(1,2)}(\mathbf{s}),$$

we conclude by (8) that

$$\begin{aligned} 1 - H_n^{(1,2)}(s_1, s_2) &\geq 1 - H_n^{(1,2)}(s_1, H_m^{(2,2)}(0)) \\ &= 1 - H_n^{(1,2)}(H_m^{(1,2)}(\mathbf{0}), H_m^{(2,2)}(0)) + O(n^{-1}) \\ &= Q_{n+m}^{(1,2)}(\mathbf{0}) + O(n^{-1}) = (1 + o(1)) c_1 (n + m)^{-1/2} + O(n^{-1}). \end{aligned}$$

Clearly, the result remains valid when  $\geq$  is replaced by  $\leq$  with  $m$  replaced by  $m - 1$ . Therefore,  $1 - H_n^{(1,2)}(s_1, s_2) \asymp n^{-1/2}$  if  $t_2 \in (0, 1]$ , and  $1 - H_n^{(1,2)}(s_1, s_2) \asymp n^{-t_2/2}$  if  $t_2 \in (1, 2)$ . This proves the first two relationships of the lemma.

Consider now the case  $t_2 \geq 2$ . In view of (6)

$$\begin{aligned} 1 - H_n^{(1,1)}(s_1) &= 1 - H_n^{(1,2)}(s_1, 1) \leq 1 - H_n^{(1,2)}(s_1, s_2) \\ &\leq 1 - H_n^{(1,1)}(s_1) + n^{-t_2} \mathbf{E}[Z_{n2} | \mathbf{Z}_0 = \mathbf{e}_1] \\ &= 1 - H_n^{(1,1)}(s_1) + (1 + o(1)) c_{12} n^{1-t_2}. \end{aligned}$$

Recalling that  $1 - s_1 = n^{-t_1}$  and selecting  $m = m(s_1)$  similar to (16) we get

$$1 - H_n^{(1,1)}(s_1) \sim 1 - H_{n+m}^{(1,1)}(0) \asymp \frac{1}{n^{t_1} + n}. \quad (17)$$

Hence, if  $t_1 < 1$  then  $1 - H_n^{(1,2)}(s_1, s_2) \asymp n^{-1}$  as claimed.

The statement for  $t_1 \geq 1, t_2 \geq 2$  follows from (11). □

### 3 Decomposable branching processes in random environment

The model of branching processes in random environment which we are dealing with is a combination of the processes introduced by Smith and Wilkinson [8] and the ordinary decomposable multitype Galton-Watson processes. To give a formal description of the model denote by  $\mathcal{M}$  the space of probability measures on  $\mathbb{N}_0^{N+1}$ , where  $\mathbb{N}_0 := \{0, 1, 2, \dots\}$  and let  $\mathbf{e}$  be a random variable with values in  $\mathcal{M}$ . An infinite sequence  $\mathcal{E} = (\mathbf{e}_1, \mathbf{e}_2, \dots)$  of i.i.d. copies of  $\mathbf{e}$  is said to form a *random environment*.

We associate with  $\mathbf{e}$  and  $\mathbf{e}_n, n = 1, 2, \dots$  random vectors  $(\xi_0, \dots, \xi_N)$  and  $(\xi_0^{(n)}, \dots, \xi_N^{(n)})$  such that for  $\mathbf{k} \in \mathbb{N}_0^{N+1}$

$$\mathbb{P}((\xi_0, \dots, \xi_N) = \mathbf{k} | \mathbf{e}) = \mathbf{e}(\{\mathbf{k}\}), \quad \mathbb{P}((\xi_0^{(n)}, \dots, \xi_N^{(n)}) = \mathbf{k} | \mathbf{e}_n) = \mathbf{e}_n(\{\mathbf{k}\}).$$

We now specify a branching process  $(X_n, \mathbf{Z}_n) = (X_n, Z_{n1}, \dots, Z_{nN})$  in random environment  $\mathcal{E}$  with types  $0, 1, \dots, N$  as follows.

- 1)  $(X_0, \mathbf{Z}_0) = (1, \mathbf{0})$ .
- 2) Given  $\mathcal{E} = (e_1, e_2, \dots)$  and  $(X_{n-1}, \mathbf{Z}_{n-1}), n \geq 1$

$$X_n = \sum_{k=1}^{X_{n-1}} \xi_{k0}^{(n-1)}, \quad Z_{nj} = \sum_{k=1}^{X_{n-1}} \xi_{kj}^{(n-1)} + \sum_{i=1}^j \sum_{k=1}^{Z_{(n-1)i}} \eta_{k,ij}^{(n-1)}, \quad j = 1, \dots, N$$



where the tuples  $(\xi_{k0}^{(n-1)}, \xi_{k1}^{(n-1)}, \dots, \xi_{kN}^{(n-1)})$ ,  $k = 1, 2, \dots, X_{n-1}$  are i.i.d. random vectors with distribution  $e_{n-1}$  i.e., given  $\mathbf{e}_{n-1} = e_{n-1}$  distributed as  $(\xi_0^{(n-1)}, \xi_1^{(n-1)}, \dots, \xi_N^{(n-1)})$ , and the tuples  $(\eta_{kii}^{(n-1)}, \eta_{ki,i+1}^{(n-1)}, \dots, \eta_{kiN}^{(n-1)})$  are independent random vectors distributed as  $(\eta_{ii}, \eta_{i,i+1}, \dots, \eta_{iN})$  for  $i = 1, 2, \dots, N$ , i.e., in accordance with the respective probability generating function  $h^{(i,N)}(\mathbf{s})$  in (4).

Informally,  $\xi_{kj}^{(n-1)}$  is the number of type  $j$  children produced by the  $k$ -th particle of type 0 of generation  $n-1$ , while  $\eta_{k,ij}^{(n-1)}$  is the number of type  $j$  children produced by the  $k$ -th particle of type  $i$  of generation  $n-1$ .

We denote by  $\mathbb{P}$  and  $\mathbb{E}$  the corresponding probability measure and expectation on the underlying probability space to distinguish them from the probability measure and expectation in constant environment specified by the symbols  $\mathbf{P}$  and  $\mathbf{E}$ .

Thus, in our model particles of type 0 belonging to the  $(n-1)$ -th generation give birth in total to  $X_n$  particles of their own type and to the tuple  $\mathbf{Y}_n = (Y_{n1}, \dots, Y_{nN})$  of daughter particles of types  $1, 2, \dots, N$ , where

$$Y_{nj} = \sum_{k=1}^{X_{n-1}} \xi_{kj}^{(n-1)}. \quad (18)$$

In particular,  $\mathbf{Y}_1 = (Y_{11}, \dots, Y_{1N}) = (\xi_1^{(0)}, \dots, \xi_N^{(0)}) = \mathbf{Z}_1$ .

Finally, each particle of type  $i = 1, 2, \dots, N$  generates its own (decomposable, if  $i < N$ ) process with  $N - i + 1$  types evolving in a constant environment.

Let  $\mu_1 = \mathbb{E}[\xi_0 | \mathbf{e}]$ ,  $\mu_2 = \mathbb{E}[\xi_0(\xi_0 - 1) | \mathbf{e}]$ , and

$$\theta_i = \mathbb{E}[\xi_i | \mathbf{e}], \quad i = 1, 2, \dots, N, \quad \Theta_1 := \sum_{l=1}^N \theta_l.$$

Our assumptions on the characteristics of the process we consider are formulated as

**Hypothesis A:**

- The initial state of the process is  $(X_0, \mathbf{Z}_0) = (1, \mathbf{0})$ ;
- particles of type 0 form (on their own) a critical branching process in a random environment, such that

$$\mathbb{E} \log \mu_1 = 0, \quad \mathbb{E} \log^2 \mu_1 \in (0, \infty); \quad (19)$$

- particles of type 0 produce particles of type 1 with a positive probability and

$$\mathbb{P}(\theta_1 > 0) = 1;$$

- particles of each type form (on their own) critical branching processes which are independent of the environment, i.e.  $m_{ii} = \mathbf{E}\eta_{ii} = 1$ ,  $i = 1, 2, \dots, N$ ;
- particles of type  $i = 1, 2, \dots, N - 1$  produce particles of type  $i + 1$  with a positive probability, i.e.,  $m_{i,i+1} = \mathbf{E}\eta_{i,i+1} > 0$ ,  $i = 1, 2, \dots, N - 1$ ;
- The second moments of the offspring numbers are finite

$$\mathbf{E}\eta_{ij}^2 < \infty, 1 \leq i \leq j \leq N \quad \text{with} \quad b_i = \frac{1}{2} \text{Var} \eta_{ii} \in (0, \infty).$$

The following theorem is the main result of the paper:

**Theorem 6** *If Hypothesis A is valid and*

$$\mathbb{E}[\mu_1^{-1}] < \infty, \quad \mathbb{E}[\mu_2 \mu_1^{-2} (1 + \max(0, \log \mu_1))] < \infty, \quad (20)$$

*then there exists a positive constant  $K_0$  such that*

$$\mathbb{P}(\mathbf{Z}_n \neq 0 | X_0 = 1, \mathbf{Z}_0 = \mathbf{0}) \sim \frac{2^{N-1} K_0}{\log n} \quad (21)$$

*and for any positive  $t_1, t_2, \dots, t_N$*

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P}\left(\frac{\log Z_{ni}}{\log n} \leq t_i, i = 1, \dots, N \mid Z_{n1} > 0\right) &= G(t_1, \dots, t_N) \\ &= 1 - \frac{1}{1 + \max(0, \min_{1 \leq l \leq N} (t_l - l))}. \end{aligned} \quad (22)$$

The proof of the theorem is divided into several stages.

Let

$$T = \min\{n \geq 0 : X_n = 0\}.$$

According to [7, Theorem 1], if conditions (19) and (20) are valid then for a positive constant  $c$

$$\mathbb{P}(X_n > 0) = \mathbb{P}(T > n) \sim \frac{c}{\sqrt{n}}, \quad n \rightarrow \infty. \quad (23)$$

Set  $S_n := \sum_{k=0}^{n-1} X_k$  and  $A_n = \max_{0 \leq k \leq n-1} X_k$ , so that  $S_T$  and  $A_T$  give the total number ever born of type 0 particles and the maximal generation size of type 0 particles.

**Lemma 7** (see [1]) *If conditions (19) and (20) are valid then there exists a constant  $K_0 \in (0, \infty)$  such that*

$$\mathbb{P}(S_T > x) \sim \mathbb{P}(A_T > x) \sim \frac{K_0}{\log x}, \quad x \rightarrow \infty. \quad (24)$$

In fact, the representation (24) has been proved in [1] under conditions (20) and (19) only for the case when the probability generating functions  $f_n(s, \mathbf{1}^{(N)})$  are linear-fractional with probability 1. However, this restriction is easily removed using the results established later on for the general case in [7] and [2].

Let now  $\|\mathbf{Y}_n\| = Y_{n1} + \dots + Y_{nN}$ ,  $\zeta_k^{(n)} = \zeta_{k1}^{(n-1)} + \dots + \zeta_{kN}^{(n-1)}$  and

$$\begin{aligned} L_{nj} &= \sum_{l=1}^n Y_{lj} = \sum_{l=1}^n \sum_{k=1}^{X_{l-1}} \zeta_{kj}^{(l-1)}, \quad B_{nj} = \max_{1 \leq l \leq n} Y_{lj}, \\ L_n &= \sum_{l=1}^n \|\mathbf{Y}_l\| = \sum_{l=1}^n \sum_{k=1}^{X_{l-1}} \zeta_k^{(l-1)}, \quad B_n = \max_{1 \leq l \leq n} \|\mathbf{Y}_l\|. \end{aligned}$$

In particular,  $L_T$  gives the total number of daughter particles of types  $1, \dots, N$  produced by type 0 particles during the evolution of the process.

**Lemma 8** *If conditions (19) and (20) are valid and  $\mathbb{P}(\Theta_1 > 0) = 1$ , then*

$$\mathbb{P}(B_T > x) \sim \mathbb{P}(L_T > x) \sim \frac{K_0}{\log x}, \quad x \rightarrow \infty. \quad (25)$$

*If conditions (20), (19) are valid and  $\mathbb{P}(\theta_j > 0) = 1$  for some  $j \in \{1, \dots, N\}$  then*

$$\mathbb{P}(B_{Tj} > x) \sim \mathbb{P}(L_{Tj} > x) \sim \frac{K_0}{\log x}, \quad x \rightarrow \infty. \quad (26)$$

PROOF For any  $\varepsilon \in (0, 1)$  we have

$$\mathbb{P}(A_T > x) \leq \mathbb{P}(B_T > x^{1-\varepsilon}) + \mathbb{P}(A_T > x; B_T \leq x^{1-\varepsilon}).$$

Let  $T_x = \min\{k : X_k > x\}$ . Then

$$\begin{aligned} \mathbb{P}(A_T > x; B_T \leq x^{1-\varepsilon}) &\leq \sum_{l=1}^{\infty} \mathbb{P}(T_x = l; \|\mathbf{Y}_{l+1}\| \leq x^{1-\varepsilon}) \\ &= \sum_{l=1}^{\infty} \mathbb{P}\left(T_x = l; \sum_{k=1}^{X_l} \zeta_k^{(l)} \leq x^{1-\varepsilon}\right) \\ &\leq \mathbb{P}(A_T > x) \mathbb{P}\left(\sum_{k=1}^{[x]} \zeta_k^{(0)} \leq x^{1-\varepsilon}\right). \end{aligned}$$

Since  $\mathbb{P}(\Theta_1 > 0) = 1$  and  $\Theta_1 = \mathbb{E}[\zeta_k^{(0)} | \mathfrak{e}]$ ,  $k = 1, 2, \dots$ , the law of large numbers gives

$$\lim_{x \rightarrow \infty} \mathbb{P} \left( \frac{1}{x\Theta_1} \sum_{k=1}^{[x]} \zeta_k^{(0)} \leq \frac{1}{x^\varepsilon \Theta_1} \middle| \mathfrak{e} \right) = 0 \quad \mathbb{P} - \text{a.s.}.$$

Thus

$$\limsup_{x \rightarrow \infty} \mathbb{P} \left( \sum_{k=1}^{[x]} \zeta_k^{(0)} \leq x^{1-\varepsilon} \right) \leq \mathbb{E} \left[ \limsup_{x \rightarrow \infty} \mathbb{P} \left( \sum_{k=1}^{[x]} \zeta_k^{(0)} \leq x^{1-\varepsilon} \middle| \mathfrak{e} \right) \right] = 0.$$

As a result, for any  $\delta > 0$  and all  $x \geq x_0(\delta)$  we get

$$(1 - \delta) \mathbb{P}(A_T > x) \leq \mathbb{P}(B_T > x^{1-\varepsilon}). \quad (27)$$

To deduce for  $\mathbb{P}(B_T > x)$  an estimate from above we write

$$\mathbb{P}(B_T > x) \leq \mathbb{P}(A_T > x^{1-\varepsilon}) + \mathbb{P}(B_T > x; A_T \leq x^{1-\varepsilon}). \quad (28)$$

Further, letting  $\hat{T}_x = \min \{k : \|\mathbf{Y}_k\| > x\}$  we have

$$\begin{aligned} \mathbb{P}(B_T > x; A_T \leq x^{1-\varepsilon}) &\leq \mathbb{P}(T > x^{\varepsilon/2}) \\ &\quad + \sum_{1 \leq l \leq x^{\varepsilon/2}} \mathbb{P}(\hat{T}_x = l; A_T \leq x^{1-\varepsilon}). \end{aligned}$$

By Markov inequality we see that

$$\begin{aligned} \sum_{1 \leq l \leq x^{\varepsilon/2}} \mathbb{P}(\hat{T}_x = l; A_T \leq x^{1-\varepsilon}) &\leq \sum_{1 \leq l \leq x^{\varepsilon/2}} \mathbb{P}(X_{l-1} \leq x^{1-\varepsilon}; \|\mathbf{Y}_l\| > x) \\ &\leq x^{\varepsilon/2} \mathbb{P} \left( \sum_{k=1}^{[x^{1-\varepsilon}]} \zeta_k^{(0)} > x \right) \leq x^{-\varepsilon/2} \mathbb{E}[\|\mathbf{Y}_1\|]. \end{aligned}$$

Hence, recalling (23) we obtain  $\mathbb{P}(B_T > x; A_T \leq x^{1-\varepsilon}) = O(x^{-\varepsilon/4})$  implying in view of (28)

$$\mathbb{P}(B_T > x) \leq \mathbb{P}(A_T > x^{1-\varepsilon}) + O(x^{-\varepsilon/4}). \quad (29)$$

Combining (27) and (29) and letting first  $x \rightarrow \infty$  and then  $\varepsilon \rightarrow 0$  justify by Lemma 7 the equivalence

$$\mathbb{P}(B_T > x) \sim \mathbb{P}(A_T > x) \sim \frac{K_0}{\log x}.$$

Finally,

$$\mathbb{P}(B_T > x) \leq \mathbb{P}(L_T > x) \leq \mathbb{P}(TB_T > x) \leq \mathbb{P}(B_T > x^{1-\varepsilon}) + \mathbb{P}(T > x^\varepsilon),$$

and applying (23) and Lemma 7 proves the first equivalence in (25).

One may check (26) by similar arguments.

□

**Corollary 9** *If conditions (19) and (20) are valid and  $\mathbb{P}(\theta_1 > 0) = 1$ , then, as  $n \rightarrow \infty$*

$$F(n) := \mathbb{E} \left[ 1 - \exp \left\{ - \sum_{i=1}^N L_{Ti} Q_n^{(i,N)}(\mathbf{0}) \right\} \right] \sim \frac{2^{N-1} K_0}{\log n}.$$

PROOF Clearly,

$$L_{T1} Q_n^{(1,N)}(\mathbf{0}) \leq \sum_{i=1}^N L_{Ti} Q_n^{(i,N)}(\mathbf{0}) \leq L_T \sum_{i=1}^N Q_n^{(i,N)}(\mathbf{0})$$

and, by (8)

$$\sum_{i=1}^N Q_n^{(i,N)}(\mathbf{0}) \sim Q_n^{(1,N)}(\mathbf{0}) \sim c_1 n^{-1/2(N-1)}.$$

To finish the proof of the corollary it remains to observe that

$$\mathbb{E} \left[ 1 - e^{-\lambda L_T} \right] \sim \mathbb{E} \left[ 1 - e^{-\lambda L_{T1}} \right] \sim \frac{K_0}{\log(1/\lambda)}, \quad \lambda \rightarrow +0, \quad (30)$$

due to Lemma 8 and the Tauberian theorem [4, Ch. XIII.5, Theorem 4] applied, for instance, to the right hand side of

$$\lambda^{-1} \mathbf{E} \left[ 1 - e^{-\lambda L_T} \right] = \int_0^\infty \mathbf{P}(L_T > x) e^{-\lambda x} dx,$$

and to use the inequalities

$$\mathbb{E} \left[ 1 - \exp \left\{ -L_{T1} Q_n^{(1,N)}(\mathbf{0}) \right\} \right] \leq F(n) \leq \mathbb{E} \left[ 1 - \exp \left\{ -L_T \sum_{i=1}^N Q_n^{(i,N)}(\mathbf{0}) \right\} \right].$$

□

PROOF of Theorem 6. We first check (21). Notice that each particle of type

$i$  of generation  $n$  has either a mother of type 0 (of generation  $n-1$ ), or an ancestor of generation  $k, 1 \leq k < n$  whose mother is of type 0; recall that the number of particles of type  $i$  of generation  $k$  having a mother of type 0 is denoted by  $Y_{ki}$ . By a decomposition of  $Z_{ni}$  based on this fact and using the branching property, we get:

$$\mathbb{E} \left[ 1 - s_1^{Z_{n1}} \dots s_N^{Z_{nN}} \right] = \mathbb{E} \left[ 1 - \prod_{k=1}^n \prod_{i=1}^N \left( H_{n-k}^{(i,N)}(\mathbf{s}) \right)^{Y_{ki}} \right] = \mathbb{E} \left[ 1 - e^{R(n;\mathbf{s})} \right],$$

where  $H_0^{(i,N)}(\mathbf{s}) = s_i$  by convention, and

$$R(n;\mathbf{s}) = \sum_{k=1}^n \sum_{i=1}^N Y_{ki} \log H_{n-k}^{(i,N)}(\mathbf{s}).$$

In particular,

$$\mathbb{P}(\mathbf{Z}_n \neq \mathbf{0}) = \mathbb{E} \left[ 1 - e^{R(n;\mathbf{0})}; T \leq \sqrt{n} \right] + O \left( \mathbb{P}(T > \sqrt{n}) \right).$$

Since  $\log(1-x) \sim -x$  as  $x \rightarrow +0$  and for  $k \leq \sqrt{n}$  and  $n \rightarrow \infty$

$$Q_n^{(i,N)}(\mathbf{0}) = 1 - H_n^{(i,N)}(\mathbf{0}) \leq Q_{n-k}^{(i,N)}(\mathbf{0}) \leq Q_{n-\sqrt{n}}^{(i,N)}(\mathbf{0}) = (1 + o(1)) Q_n^{(i,N)}(\mathbf{0}),$$

we obtain

$$\begin{aligned} \mathbb{E} \left[ e^{R(n;\mathbf{0})}; T \leq \sqrt{n} \right] &= \mathbb{E} \left[ \exp \left\{ - (1 + o(1)) \sum_{i=1}^N L_{ni} Q_n^{(i,N)}(\mathbf{0}) \right\}; T \leq \sqrt{n} \right] \\ &= \mathbb{E} \left[ \exp \left\{ - (1 + o(1)) \sum_{i=1}^N L_{Ti} Q_n^{(i,N)}(\mathbf{0}) \right\}; T \leq \sqrt{n} \right] \\ &= \mathbb{E} \left[ \exp \left\{ - (1 + o(1)) \sum_{i=1}^N L_{Ti} Q_n^{(i,N)}(\mathbf{0}) \right\} \right] - O \left( \mathbb{P}(T > \sqrt{n}) \right). \end{aligned}$$

Thus,

$$\mathbb{P}(\mathbf{Z}_n \neq \mathbf{0}) = \mathbb{E} \left[ 1 - \exp \left\{ - (1 + o(1)) \sum_{i=1}^N L_{Ti} Q_n^{(i,N)}(\mathbf{0}) \right\} \right] + O \left( \mathbb{P}(T > \sqrt{n}) \right), \quad (31)$$

and (21) follows from Corollary 9 and (23).

Now we prove (22). Recall that we always take  $X_0 = 1, \mathbf{Z}_0 = \mathbf{0}$ .

Consider first the case  $N = 1$ . Writing for simplicity  $Y_k = Y_{k1}, Z_n = Z_{n1}$ ,  $s = s_1$  and  $H_n(s) = H_n^{(1,1)}(s) = \mathbf{E} [s^{Z_n} | Z_0 = 1]$  we have

$$\mathbf{E} [s^{Z_n} | Z_n > 0] = \frac{\mathbf{E} [s^{Z_n}] - \mathbf{E} (Z_n = 0)}{\mathbb{P} (Z_n > 0)} = 1 - \frac{\mathbf{E} [1 - s^{Z_n}]}{\mathbb{P} (Z_n > 0)},$$

and by (31)

$$\mathbf{E} [1 - s^{Z_n}] = \mathbf{E} \left[ 1 - \exp \left\{ \sum_{k=1}^n Y_k \log H_{n-k}(s) \right\} \right].$$

By the criticality condition  $1 - H_n(0) \sim (b_1 n)^{-1}$ . Thus, if  $s = e^{-\lambda/(b_1 n^t)}$ , then

$$1 - s \sim \lambda / (b_1 n^t) \sim 1 - H_{[n^t/\lambda]}(0),$$

where  $[x]$  denotes the integral part of  $x$ . Hence it follows that for any  $t > 1$  as  $n \rightarrow \infty$

$$1 - H_n(e^{\lambda/n^t}) \sim 1 - H_n(H_{[n^t/\lambda]}(0)) = 1 - H_{n+[n^t/\lambda]}(0) \sim \lambda / (b_1 n^t).$$

This, similar to the previous estimates for the survival probability of the  $(N + 1)$ -type branching process gives (recall that  $(X_0, Z_0) = (1, 0)$ )

$$\mathbf{E} [1 - \exp \{-\lambda Z_n / (b_1 n^t)\}] \sim \mathbf{E} [1 - \exp \{-\lambda c n^{-t} L_{T1}\}] \sim \frac{K_0}{t \log n}.$$

Since  $\mathbf{P}(Z_n > 0) \sim K_0 / \log n$ , it follows that for any fixed  $t > 1$  and  $\lambda > 0$

$$\lim_{n \rightarrow \infty} \mathbf{E} [\exp \{-\lambda Z_n / (b_1 n^t)\} | Z_n > 0] = 1 - \frac{1}{t}.$$

This implies that the conditional law of  $Z_n / (b_1 n^t)$  given  $Z_n > 0$  converges to the law of a random variable  $X$  with  $\mathbf{P}(X = 0) = 1 - t^{-1}$  and  $\mathbf{P}(X = +\infty) = t^{-1}$ . Therefore, for any  $t > 1$

$$\begin{aligned} G(t) &= \lim_{n \rightarrow \infty} \mathbb{P} (n^{-t} Z_n \leq b_1 | Z_n > 0) \\ &= \lim_{n \rightarrow \infty} \mathbb{P} \left( \frac{\log Z_n}{\log n} \leq t \mid Z_n > 0 \right) = 1 - \frac{1}{t}. \end{aligned} \quad (32)$$

Since  $\lim_{t \downarrow 1} G(t) = 0$  we may rewrite (32) for any  $t > 0$  as

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( \frac{\log Z_n}{\log n} \leq t \mid Z_n > 0 \right) = 1 - \frac{1}{1 + \max(0, t - 1)}, \quad (33)$$

as desired.

Now we consider the case  $N \geq 2$  and use the equality

$$\begin{aligned} \mathbb{E} \left[ s_1^{Z_{n1}} \dots s_N^{Z_{nN}} | Z_{n1} > 0 \right] &= \frac{\mathbb{E} \left[ 1 - s_2^{Z_{n2}} \dots s_N^{Z_{nN}} I \{ Z_{n1} = 0 \} \right]}{\mathbb{P}(Z_{n1} > 0)} \\ &\quad - \frac{\mathbb{E} \left[ 1 - s_1^{Z_{n1}} \dots s_N^{Z_{nN}} \right]}{\mathbb{P}(Z_{n1} > 0)}. \end{aligned} \quad (34)$$

We study each term at the right-hand side of (34) separately. By (31) and  $\log(1-x) \sim -x, x \rightarrow +0$  we see that, as  $n \rightarrow \infty$

$$\mathbb{E} \left[ 1 - s_1^{Z_{n1}} \dots s_N^{Z_{nN}} \right] = \mathbb{E} \left[ 1 - \exp \{ -(1 + o(1)) R_N(n, \mathbf{s}) \} \right], \quad (35)$$

where

$$R_N(n, \mathbf{s}) := \sum_{k=1}^n \sum_{i=1}^N Y_{ki} Q_{n-k}^{(i,N)}(\mathbf{s}).$$

Let now  $t_1, \dots, t_N$  be a tuple of positive numbers satisfying (10). It follows from Lemma 3 that, for  $1 - s_l = n^{-t_l}, l = 1, \dots, N$

$$Q_n^{(i,N)}(\mathbf{s}) \asymp n^{-\min_{i \leq l \leq N} (t_l - l + i)} = n^{-i - \min_{i \leq l \leq N} (t_l - l)}. \quad (36)$$

Since

$$\min_{1 \leq i \leq N} \min_{i \leq l \leq N} (t_l - l + i) = \min_{1 \leq l \leq N} (t_l - l + 1) \geq 1 \quad (37)$$

by our conditions, we have as  $n \rightarrow \infty$ :

$$Q_n^{(i,N)}(\mathbf{s}) \ll Q_n^{(1,N)}(\mathbf{s}) \asymp n^{-\min_{1 \leq l \leq N} (t_l - l + 1)}.$$

Thus, there exist constants  $C_j, j = 1, 2, 3, 4$  such that, on the set  $T \leq \sqrt{n}$  the estimates

$$C_1 L_{T1} Q_n^{(1,N)}(\mathbf{s}) \leq R_N(n, \mathbf{s}) \leq \sum_{k=1}^n \sum_{i=1}^N Y_{ki} Q_{n-k}^{(i,N)}(\mathbf{s}) \leq C_2 L_T \sum_{i=1}^N Q_n^{(i,N)}(\mathbf{s})$$

are valid for all sufficiently large  $n$ . This, in turn, implies

$$C_3 L_{T1} n^{-\min_{1 \leq l \leq N} (t_l - l + 1)} \leq R_N(n, \mathbf{s}) \leq C_4 n^{-\min_{1 \leq l \leq N} (t_l - l + 1)} L_T. \quad (38)$$



Using the estimates above and (30) we get for the selected  $t_1, \dots, t_N$ , as  $n \rightarrow \infty$

$$\begin{aligned} \mathbb{E} \left[ 1 - \exp \{ -R_N(n, \mathbf{s}) \} ; T \leq \sqrt{n} \right] &= \frac{1}{\log n} \frac{(1 + o(1)) K_0}{1 + \min_{1 \leq l \leq N} (t_l - l)} \\ &\quad + O \left( \mathbb{P} (T > \sqrt{n}) \right), \end{aligned}$$

which leads on account of (23) to

$$\lim_{n \rightarrow \infty} (\log n) \mathbb{E} \left[ 1 - s_1^{Z_{n1}} \dots s_N^{Z_{nN}} \right] = \frac{K_0}{1 + \min_{1 \leq l \leq N} (t_l - l)}. \quad (39)$$

Thus,

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E} \left[ 1 - s_1^{Z_{n1}} \dots s_N^{Z_{nN}} \right]}{\mathbb{P} (Z_{n1} > 0)} = \frac{1}{1 + \min_{1 \leq l \leq N} (t_l - l)} < 1.$$

Further,

$$\mathbb{E} \left[ 1 - s_2^{Z_{n2}} \dots s_N^{Z_{nN}} I \{ Z_{n1} = 0 \} \right] = \mathbb{E} \left[ 1 - \exp \left\{ \sum_{k=1}^n \sum_{i=1}^N Y_{ki} \log H_{n-k}^{(i,N)}(0, \mathbf{s}_2) \right\} \right].$$

By definitions of  $H_n^{(i,N)}(\mathbf{s})$ , estimates (36) and the choice of  $s_i, i = 2, \dots, N$  we have

$$1 - H_n^{(i,N)}(0, \mathbf{s}_2) = 1 - H_n^{(i,N)}(\mathbf{s}) = Q_n^{(i,N)}(\mathbf{s}) \asymp n^{-\min_{i \leq l \leq N} (t_l - l + i)} = o(n^{-1}).$$

Besides, as  $n \rightarrow \infty$

$$1 - H_n^{(1,N)}(0, \mathbf{s}_2) = Q_n^{(1,N)}(0, \mathbf{s}_2) \sim c_1 n^{-1} \quad (40)$$

by Lemma 4. Hence it follows that on the set  $T \leq \sqrt{n}$ ,

$$\begin{aligned} \sum_{k=0}^{T-1} \sum_{i=1}^N Y_{ki} \log H_{n-k}^{(i,N)}(0, \mathbf{s}_2) &= -(1 + o(1)) \sum_{k=0}^{T-1} \sum_{i=1}^N Y_{ki} Q_{n-k}^{(i,N)}(0, \mathbf{s}_2) \\ &= -(1 + o(1)) \sum_{i=1}^N L_{Ti} Q_n^{(i,N)}(0, \mathbf{s}_2) \end{aligned}$$

and, moreover,

$$Q_n^{(1,N)}(0, \mathbf{s}_2) L_{T1} \leq \sum_{i=1}^N L_{Ti} Q_n^{(i,N)}(0, \mathbf{s}_2) \leq C_2 Q_n^{(1,N)}(0, \mathbf{s}_2) L_T.$$

Using now the same line of arguments as earlier one may show that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ 1 - s_2^{Z_{n2}} \dots s_N^{Z_{nN}} I \{Z_{n1} = 0\} \right] \log n = K_0,$$

implying by (21) with  $N = 1$  that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E} \left[ 1 - s_2^{Z_{n2}} \dots s_N^{Z_{nN}} I \{Z_{n1} = 0\} \right]}{\mathbb{P}(Z_{n1} > 0)} = 1.$$

As a result, given (10) we have

$$G(t_1, \dots, t_N) = \lim_{n \rightarrow \infty} \mathbb{E} \left[ s_1^{Z_{n1}} \dots s_N^{Z_{nN}} | Z_{n1} > 0 \right] = 1 - \frac{1}{1 + \min_{1 \leq l \leq N} (t_l - l)}.$$

Since  $\lim_{\min_{1 \leq l \leq N} (t_l - l) \downarrow 0} G(t_1, \dots, t_N) = 0$  we conclude by the same arguments that have been used to derive (32) and (33) that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ s_1^{Z_{n1}} \dots s_N^{Z_{nN}} | Z_{n1} > 0 \right] = 1 - \frac{1}{1 + \max(0, \min_{1 \leq l \leq N} (t_l - l))}$$

for all positive  $t_1, \dots, t_N$ , completing the proof of Theorem 6.  $\square$

## 4 The case of three types

It follows from (8) that for a strongly critical  $N$ -type decomposable branching process in a constant environment

$$\mathbf{P}(\mathbf{Z}_n \neq \mathbf{0} | \mathbf{Z}_0 = \mathbf{e}_1) \sim \mathbf{P}(Z_{n1} + \dots + Z_{n,N-1} = 0, Z_{nN} > 0 | \mathbf{Z}_0 = \mathbf{e}_1).$$

Thus, given the condition  $\{\mathbf{Z}_n \neq \mathbf{0}\}$  we observe in the limit, as  $n \rightarrow \infty$  only type  $N$  particles. This is not the case for the strongly critical  $(N + 1)$ -type decomposable branching process in a random environment. We justify this claim by considering a strongly critical branching process with three types and prove the following statement.

**Theorem 10** *Let  $N = 2$ . If hypothesis A is valid then*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( \frac{\log Z_{n1}}{\log n} \leq t_1, \frac{\log Z_{n2}}{\log n} \leq t_2 \middle| \mathbf{Z}_n \neq \mathbf{0}, X_0 = 1, \mathbf{Z}_0 = \mathbf{0} \right) = A(t_1, t_2), \quad (41)$$

where

$$A(t_1, t_2) = \begin{cases} 0, & \text{if } t_1 \in [0, \infty), \quad 0 \leq t_2 \leq 1; \\ 1 - t_2^{-1}, & \text{if } t_1 \in [0, \infty), \quad 1 < t_2 < 2; \\ 1/2, & \text{if } 0 \leq t_1 < 1, \quad t_2 \geq 2; \\ 1 - \frac{1}{2} \frac{1}{1 + \min(t_1 - 1, t_2 - 2)}, & \text{if } t_1 \geq 1, \quad t_2 \geq 2. \end{cases}$$

**Remark 11** Since the survival probability of particles of type 0 up to moment  $n$  is of order  $n^{-1/2}$ , particles of this type are absent in the limit.

**Remark 12** Since  $\lim_{\min(t_1, t_2 - 1) \downarrow 0} A(t_1, t_2) = 0$ , Theorem 10 gives a complete description of the limiting distribution for the left-hand side of (41).

PROOF of Theorem 10. We have

$$\mathbb{E} \left[ s_1^{Z_{n1}} s_2^{Z_{n2}} | \mathbf{Z}_n \neq \mathbf{0} \right] = 1 - \frac{\mathbb{E} \left[ 1 - s_1^{Z_{n1}} s_2^{Z_{n2}} \right]}{\mathbb{P}(\mathbf{Z}_n \neq \mathbf{0})},$$

where

$$\mathbb{E} \left[ 1 - s_1^{Z_{n1}} s_2^{Z_{n2}} \right] = \mathbb{E} \left[ 1 - \exp \left\{ \sum_{k=1}^n \sum_{i=1}^2 Y_{ki} \log H_{n-k}^{(i,N)}(\mathbf{s}) \right\} \right].$$

Let now  $1 - s_i = n^{-t_i}$ . If  $t_1 \geq 1$  and  $t_2 \geq 2$  then by (21) (with  $N = 2$ ) and (39) we have

$$A(t_1, t_2) = 1 - \lim_{n \rightarrow \infty} \frac{\mathbb{E} \left[ 1 - s_1^{Z_{n1}} s_2^{Z_{n2}} \right]}{\mathbb{P}(\mathbf{Z}_n \neq \mathbf{0})} = 1 - \frac{1}{2} \frac{1}{1 + \min(t_1 - 1, t_2 - 2)},$$

proving Theorem 10 for  $\min(t_1 - 1, t_2 - 2) \geq 0$ . Observe that

$$\lim_{\min(t_1 - 1, t_2 - 2) \downarrow 0} A(t_1, t_2) = 1/2,$$

and, therefore, contrary to the case  $\mathbb{P}(Z_{n1} > 0)$  we need to analyze the case of positive  $t_1, t_2$  meeting the condition  $\min(t_1 - 1, t_2 - 2) < 0$  in more detail.

The same as in the proof of Theorem 6, it is necessary to obtain estimates from above and below for

$$R_2(n, \mathbf{s}) = \sum_{k=1}^n \sum_{i=1}^2 Y_{ki} Q_{n-k}^{(i,2)}(\mathbf{s})$$

given  $T \leq \sqrt{n}$ . Observe that in view of Lemma 5 and the representation

$$Q_n^{(2,2)}(s_2) = 1 - H_n^{(2,2)}(s_2) \asymp \frac{1}{n^{t_2} + n},$$

we have

$$1 - H_n^{(1,2)}(s_1, s_2) + 1 - H_n^{(2,2)}(s_2) \asymp 1 - H_n^{(1,2)}(s_1, s_2) = Q_n^{(1,2)}(s_1, s_2).$$

This, in turn, yields for  $T \leq \sqrt{n}$

$$C_1 Q_n^{(1,2)}(s_1, s_2) L_{T1} \leq R_2(n, \mathbf{s}) \leq C_2 Q_n^{(1,2)}(s_1, s_2) L_T.$$

From this estimate, (30) and Lemma 5 we get as  $n \rightarrow \infty$

$$\mathbb{E} \left[ 1 - s_1^{Z_{n1}} s_2^{Z_{n2}} \right] \sim \frac{K_0}{C(t_1, t_2)} \log n,$$

where

$$C(t_1, t_2) = \begin{cases} 1/2 & \text{if } t_1 \in (0, \infty), 0 < t_2 \leq 1; \\ t_2/2 & \text{if } t_1 \in (0, \infty), 1 < t_2 < 2; \\ 1 & \text{if } 0 < t_1 < 1, t_2 \geq 2; \\ 1 + \min(t_1 - 1, t_2 - 2) & \text{if } t_1 \geq 1, t_2 \geq 2. \end{cases}$$

Since  $\mathbb{P}(\mathbf{Z}_n \neq \mathbf{0}) \sim 2K_0 (\log n)^{-1}$  for  $N = 2$ , we conclude that for positive  $t_1$  and  $t_2$

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E} \left[ s_1^{Z_{n1}} s_2^{Z_{n2}} \middle| \mathbf{Z}_n \neq \mathbf{0}, X_0 = 1, \mathbf{Z}_0 = \mathbf{0} \right] &= 1 - \lim_{n \rightarrow \infty} \frac{\mathbb{E} \left[ 1 - s_1^{Z_{n1}} s_2^{Z_{n2}} \right]}{\mathbb{P}(\mathbf{Z}_n \neq \mathbf{0})} \\ &= 1 - \frac{1}{2C(t_1, t_2)} = A(t_1, t_2). \end{aligned}$$

Hence the statement of Theorem 10 follows in an ordinary way.  $\square$

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